Percolation, renormalization, and quantum-computing with non-deterministic gates

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We apply a notion of static renormalization to the preparation of entangled states for quantum computing, exploiting ideas from percolation theory. Such a strategy yields a novel way to cope with the randomness of non-deterministic quantum gates. This is most relevant in the context of linear optical architectures, where probabilistic gates are inevitable. We demonstrate how to efficiently construct cluster states without the need for rerouting, thereby avoiding a massive amount of feed-forward and conditional dynamics, and furthermore show that except for a single layer of fusion measurements during the preparation, all further measurements can be shifted to the final adapted single qubit measurements. Remarkably, the cluster state preparation is achieved using essentially the same scaling in resources as if deterministic gates were available. [1]

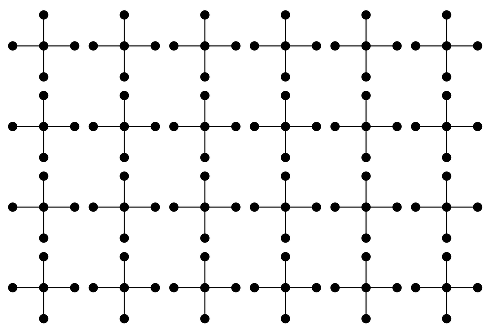
Cluster state generation

- How to produce cluster states with probabilistic gates efficiently is known [2, 3, 4].
- These schemes makes extensive use of conditional rerouting: Interactions between any two qubits must be allowed for (hard to achieve interferometric stability) and all qubits have to be stored during feed-forward (good quantum memory needed).
- Therefore they are not so well-suited when it comes to some implementations: optical lattices (only nearest neighbors), linear optics (no quantum memory), ...
- One step towards experimental feasibility: ban re-routing.

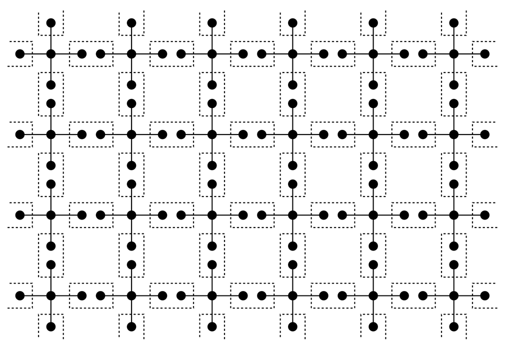
Problem: Is there a scheme to produce cluster states with a static setup and only nearest neighbor interactions?

Static setups

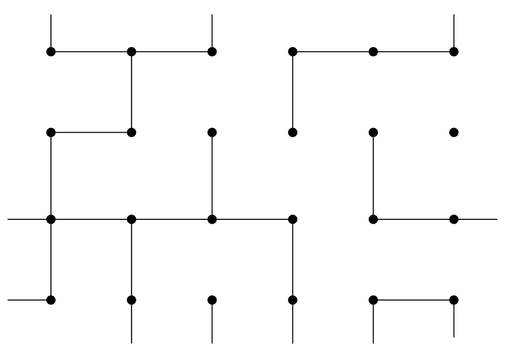
Align basic resources (single qubits, star clusters, etc.) on a lattice.



Apply non-deterministic, but heralded, entangling gates between nearest neighbors. For example: parity-check in linear optics, nearest neighbor interaction in optical lattices with non-unit filling factor, entangling distant atoms.



Yield some randomly connected graph state. Although some bonds and sites might not be existent, we know of any failures and on-site defects.



Besides looking through the results of an instance *a posteriori*, what statements on the existance of a useful graph state can be made *a priori*?

Problem: How to ensure that a cluster of a certain size can be "distilled" by local measurements from this percolated lattice?

Basics in percolation theory [5]

- *Bond percolation* on a lattice creates edges between neighboring vertices with a probability of $p < 1$. It is the same for all edges and all these processes are independent.
- For any lattice in $d \geq 2$ dimensions there exists a *critical probability* $p^{(c)}$ such that

$$\lim_{L\to\infty} P\begin{pmatrix}\text{there exists a connection be-}\\ \text{tween two opposite faces}\end{pmatrix} = 1$$

if $p > p^{(c)}$ where L is the size of the lattice. Such a connection is called a *spanning cluster*.
- Events that become more likely as p grows are called *increasing events*. The *FKG inequality* states that any pair of such events, X and Y, is positively correlated:

$$P(A \cap B) \geq P(A)P(B)$$

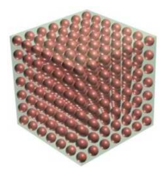
Resource scaling without rerouting

- Only allow for static setups on a lattice with neighboring sites being connected with $p > 0$.
- Fix a desired overall success probability $P < 1$.
- Chose some $\varepsilon > 0$.

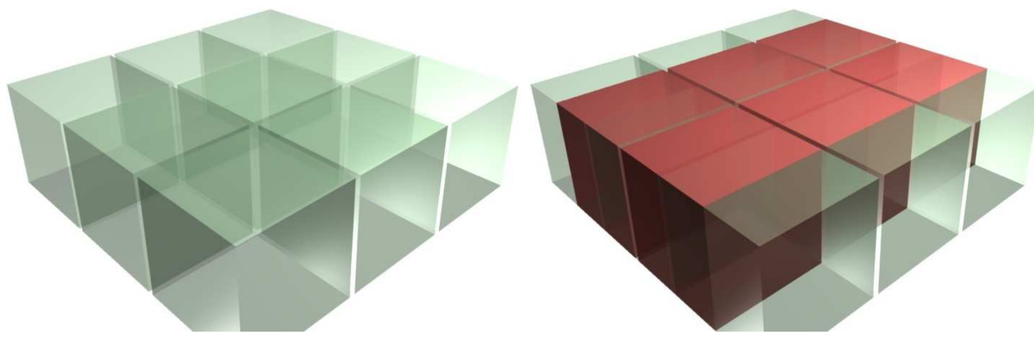
Then, the number of qubits needed scale as

$$O\left(L^{2+\varepsilon}\right)$$

Events

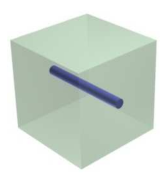


Given L^2 blocks A_{x_1,x_2} of size $k \times k \times k$, align them on a square lattice with $x_1 \in [1, L]$ and $x_2 = 2, 4, \ldots, 2L$. Define blocks B_{x_1,x_2}, $x_1 \in [1, L]$ and $x_2 = 3, 5, \ldots, 2L-1$, shifted by $k/2$ in the second dimension:



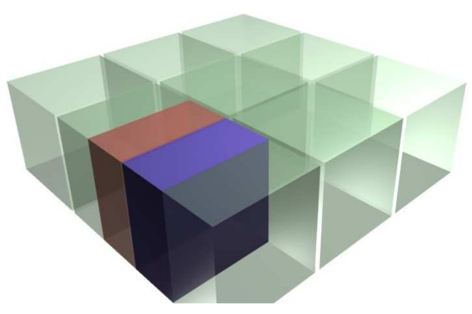
The event we are looking for (fully renormalized lattice) can be built by suitable combinations of the following events:

- W_x: Front-back spanning cluster exists in A_x. $P(W_x) \geq 1 - \mathrm{e}^{-\gamma k^2}$



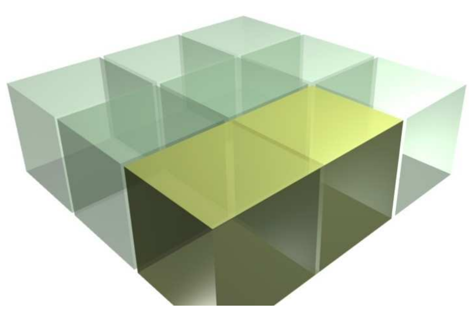
- X_x: Front-back spanning cluster exists in B_x. $P(X_x) \geq 1 - \mathrm{e}^{-\gamma k^2}$
- Y_x: At most one front-back spanning cluster in $B_{x_1,x_2} \cap A_{x_1,x_2\pm 1}$ [6].

$$P(Y_x) \geq 1 - k^6 a \mathrm{e}^{-ck}$$



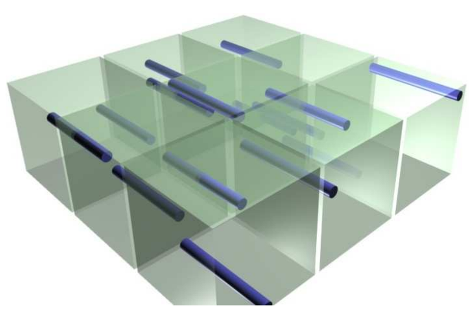
- Z_x: At most one front-back spanning cluster in $A_{x_1,x_2} \cap A_{x_1+1,x_2}$.

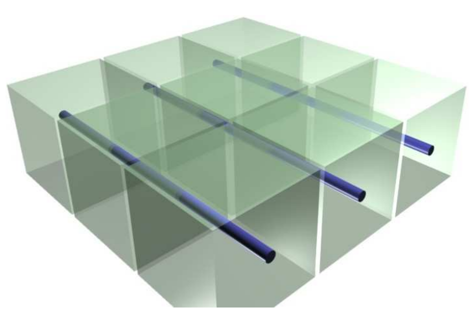
$$P(Z_x) \geq 1 - (2k)^6 a \mathrm{e}^{-2ck}$$



Proof

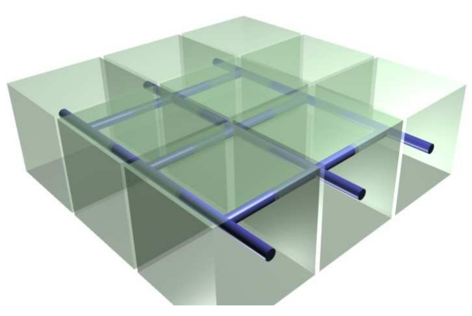
In the case of the event $X = \left(\bigcap_x W_x\right) \cap \left(\bigcap_x X_x\right)$ spanning clusters exist in all of the blocks.





If $Y = X \cap \left(\bigcap_x Y_x\right)$ happens, these spanning clusters must be connected in the front-back direction.

For $Z = Y \cap \left(\bigcap_x Z_x\right)$ each of the A_{x_1,x_2} is connected to $A_{x_1\pm 1,x_2}$, so there exists a fully connected $L \times L$ lattice of spanning clusters.



The overall probability of this event reads

$$P(Z) = P\left(\left(\bigcap_x W_x\right) \cap \left(\bigcap_x X_x\right) \cap \left(\bigcap_x Y_x\right) \cap \left(\bigcap_x Z_x\right)\right)$$

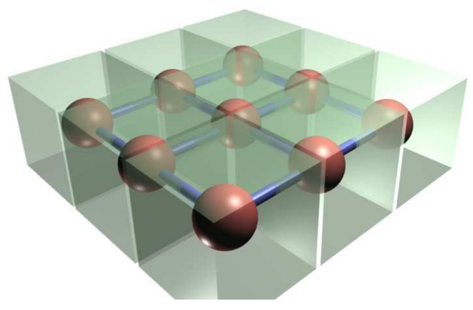
Now, W, X, Y, and Z events are pairwise positively correlated. Application of the FKG inequality yields

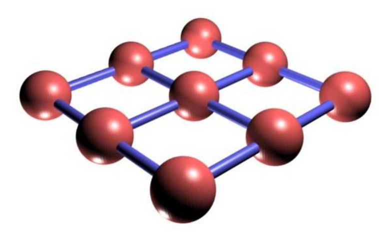
$$P(Z) \geq P_p(L,k) = \left(\prod_x P(W_x)\right)\left(\prod_x P(X_x)\right)\left(\prod_x P(Y_x)\right)\left(\prod_x P(Z_x)\right)$$

Given p and P, choosing $k(L) = L^\mu$ for any fixed $\mu > 0$ ensures $P_p(L, k(L)) \geq P$ as $L \to \infty$.

Getting rid of the rest

Use an efficient classical algorithm to identify the renormalized qubits and bonds.



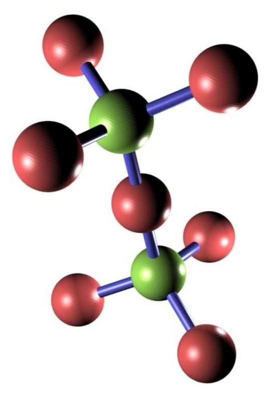


Measurements to reduce the state to the desired cluster state are postponed.

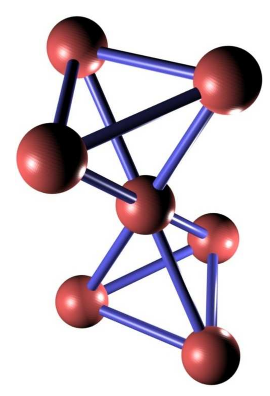
Because a bunch of measurements will follow for the sake of one-way computing anyway, measurements to reduce the percolated lattice to a fully connected renormalized cluster state can be incorporated there.

Dangling tree-like structures might be exploited for increasing the loss tolerance after the fusion step [7].

Decreasing the resource size



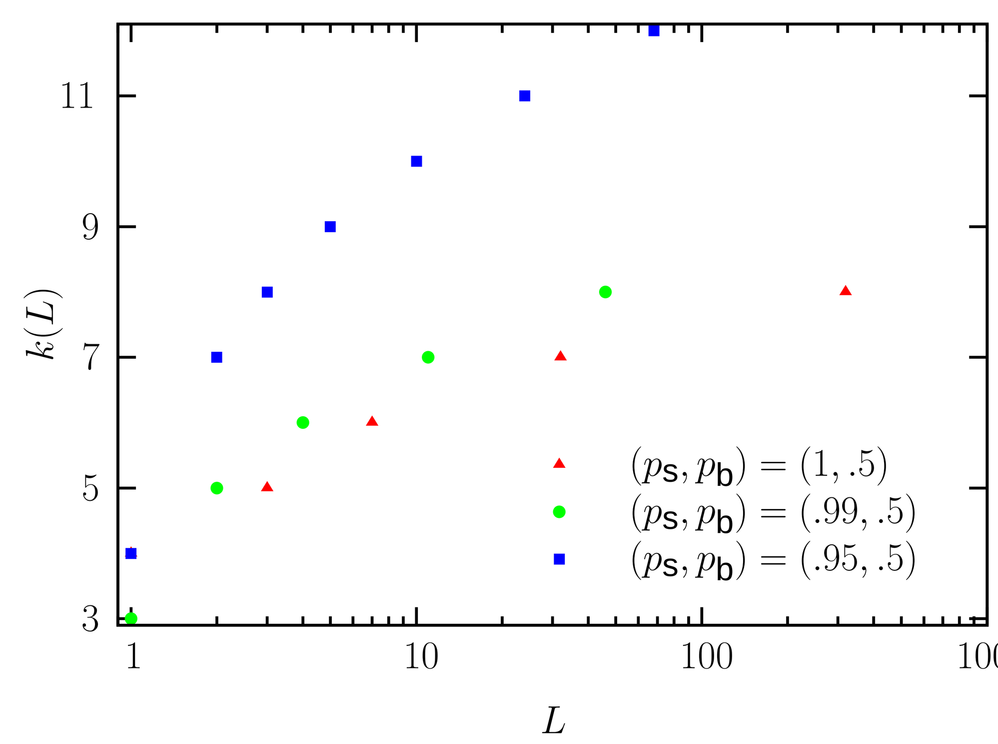
Use the covering lattice: The central qubit of "star" resources is irrelevant as long as all paths between all its neighbors still exist.



- Simultaneous fusion: If the gates used have suitable error outcomes, they might be applied in parallel without the need of feed-forward. This technique increases the effective probabilities of the gates.
- A suitable lattice for linear optics gates ($p \leq 1/2$) is the diamond lattice, using 4-qubit resources.

Numerical simulations

Monte-Carlo simulations using the Hoshen-Koppelman-Algorithm [8] have been carried out on diamond lattices with bond (p_b) and site (p_s) percolation.



Even a reasonable amount of losses per site (errors during generation of stars, on-site defects) can be dealt with. As long as $(p_\mathrm{s}, p_\mathrm{b}) > (p_\mathrm{s}, p_\mathrm{b})^{(c)}$ a sub-logarithmic scaling of $k(L)$ can be observed.

References

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