# Minimal resources for linear optical one-way computing

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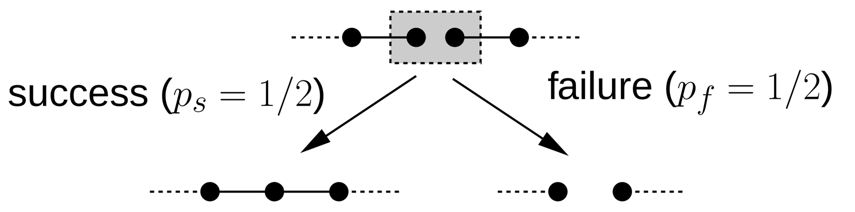
We address the question of how many maximally entangled pairs are needed in order to build up cluster states for quantum computing using the toolbox of linear optics. As the needed gates in dual-rail encoding are necessarily probabilistic with known optimal success probability, this question amounts to finding the optimal strategy for building up cluster states, from the perspective of classical control. We develop a notion of classical strategies, and present rigorous statements on the ultimate maximal and minimal use of resources of the globally optimal strategy. We find that this strategy – being also the most robust with respect to decoherence – gives rise to an advantage of already more than an order of magnitude in the number of maximally entangled pairs when building chains with an expected length of $L = 40$, compared to other legitimate strategies. For two-dimensional cluster states, we present a first scheme achieving the optimal quadratic asymptotic scaling. This analysis shows that the choice of appropriate classical control leads to a very significant reduction in resource consumption. [1, 2, 3]

## Linear optics fusion gates

- Non-unit success probabilities are inherent in linear optics [4, 5] ⇒ Huge resource overhead for near-deterministic outcome.
- *Cluster states* [6] for universal quantum computing may reduce the required overhead significantly.
- We use type-I *fusion gates* (parity check) [7, 8] to build linear cluster states from elementary EPR pairs:




- Optimal success probability of parity check without additional photons: $p_s \le 1/2$ known from results on Bell state measurements [4].

Problem: Given the optimal $p_s$, how can one optimally exploit the remaining freedom of choice concerning the *classical control*?

## Configurations and strategies

*Configuration* $C$: A set of linear clusters.

$$C = \{\bullet\!-\!\bullet\!-\!\bullet, \bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet, \cdots\}$$

*Strategy* $S$: Prescription, given a configuration, which chains shall be fused together.

$$SC = \frac{1}{2}\{\bullet\!-\!\bullet, \bullet\!-\!\bullet\!-\!\bullet, \cdots\} + \frac{1}{2}\{\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet, \cdots\}$$

*Configuration space* $\mathcal{C}_N$: All possible configurations of an overall length $\le N$.

- Strategies are defined on $\mathcal{C}_N$ which grows rapidly [9]:

$$|\mathcal{C}_N| = \frac{1 + O(N^{-1/6})}{\sqrt{8\pi^2 N}} e^{\pi\sqrt{\frac{2N}{3}}}$$

- After applying a strategy $S$ again and again, only one single chain is left.
- Expected length of that chain is denoted by $\tilde{Q}_S(N)$. We define the *quality* of $N$ EPR pairs by

$$Q(N) := \sup_S \tilde{Q}_S(N)$$

## Example strategies

### Greed

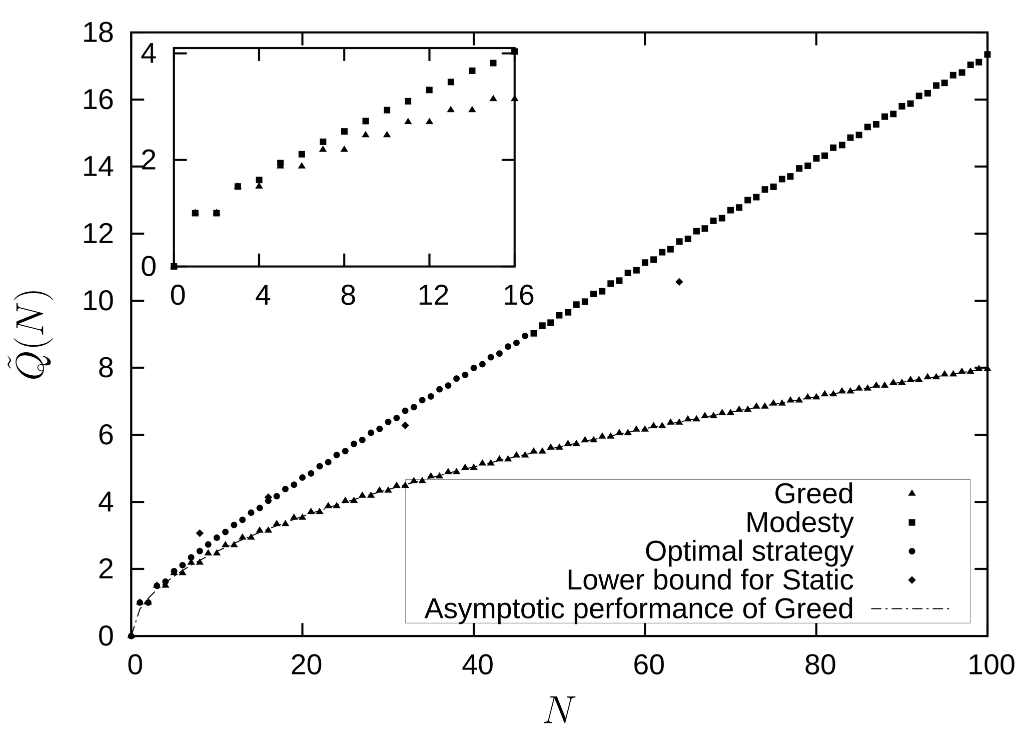
- Try to fuse together the longest chains.
- Equivalent to a one-dimensional random walk with reflecting wall. It can be solved approximately for large $N$, delivering

$$\tilde{Q}_G(N) \sim \sqrt{\frac{2N}{\pi}}$$

- $\tilde{Q}_G(N)$ is very cheap to calculate, even explicitly.

### Modesty

- Minimize the risk: try to fuse together the shortest chains.
- $\tilde{Q}_M(N)$ behaves already qualitatively better than $\tilde{Q}_G(N)$.
- We calculated $\tilde{Q}_M(N)$ for $N \le 184$.




## Optimal strategy

Problem: Find the optimal strategy, or $Q(N)$.

- A backtracking algorithm was implemented to find $Q(N)$ with an effort of

$$O\left(|\mathcal{C}_N| \left(\log|\mathcal{C}_N|\right)^5\right)$$

- Using this, we know the optimal strategy for $N \le 46$.

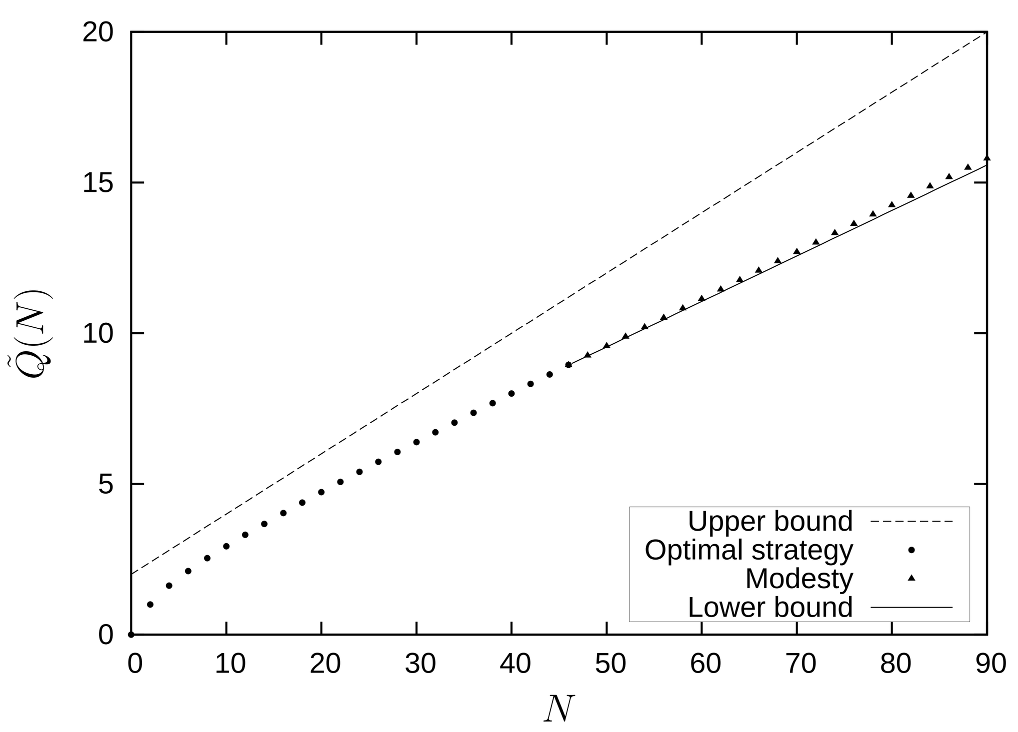
## Lower bound

- Any strategy $S$ may be used (at most optimal).
- Reservoir of EPR pairs is divided into blocks of a size between $N_0$ and $2N_0$ where $\tilde{Q}_S$ is known from computation.
- The bound

$$Q(N) \ge \tilde{Q}_S(N_0) + \alpha\,(N - N_0), \qquad \alpha = \frac{\tilde{Q}_S(N_0) - 2}{N_0}$$

can be made as tight as calculations allow for. We used

$$N_0 = 92 \quad\Longrightarrow\quad \tilde{Q}_M(N_0) = 16.1069, \quad \alpha = 0.153336$$



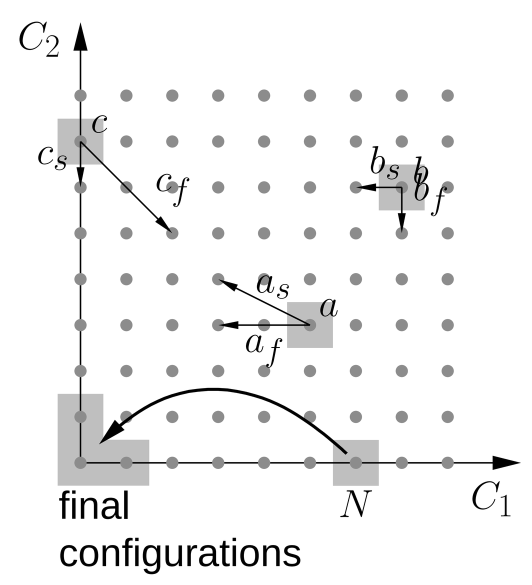

## Upper bound

- Besides the expected length, the *expected number of tries* $\langle T\rangle$ (average number of fusion attempts) can be used:

$$\tilde{Q} = N - \langle T\rangle$$

*Razor model*: Cut after every step the length of the chains down to the razor parameter $R$.

- Configuration space simplifies enormously: For $R = 2$ it is only $\mathbb{N} \times \mathbb{N}$, i. e. it can be interpreted as 2-D random walk.
- Expected number of tries $\langle T\rangle$ for the full problem can be bounded by the number of tries in the razor model.



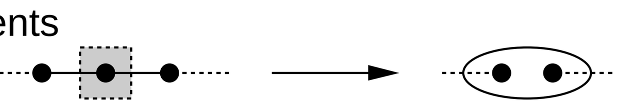

- Very few operations are left: For $R = 2$ only $a$ (fuse two EPR pairs), $b$ (fuse two GHZ states) and $c$ (fuse one EPR with one GHZ state).
- The objective of a strategy now reads: Reach the final configurations (at most one chain left) using $a$, $b$, and $c$.
- What is the minimum expected number of tries? ⇒ This problem can be cast into a *linear program*.
- Using the *dual problem*, one attains the bound

$$Q(N) \le 2 + \frac{N}{5}$$

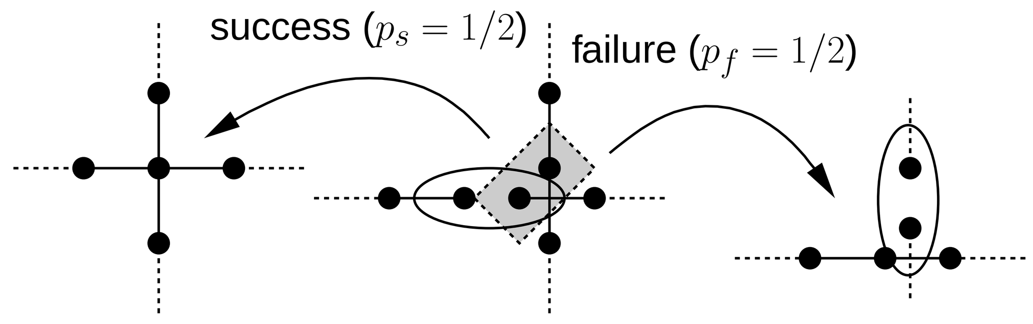
At least **five** EPR pairs are needed **to add one** edge to the chain on average!

## 2-D cluster fusion

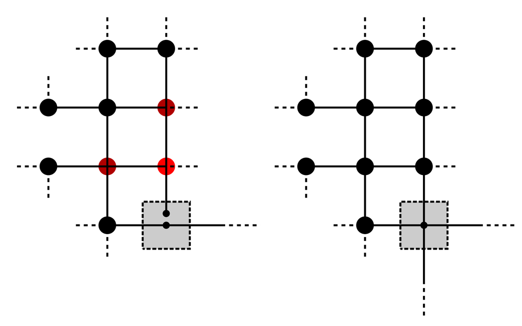
- Ingredients: 1-D cluster chains
- Tools: $\sigma_x$ measurements



and type-II fusion gates [8]



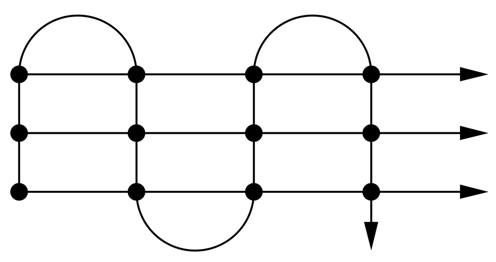

- More involved than 1-D: No order can be imposed (like the length of a chain) to assess a 2-D cluster.
- When "free ends" are not long enough, fusion (▢) failures may produce situations where no further growth is possible. • and • are endangered when trying to enlarge the overhead afterwards, destroying them *irrevocably* may be a result.



Modified problem: Given an *overall success probability* $P_s$, how does the *required overhead* scale to build a cluster of size $n \times n$?

## Optimal 2-D scaling

- A suitable *weaving pattern* designed for simple calculation of $P_s$ in terms of the chains' lengths $m$:



$$P_s = \left(\frac{1}{2^m}\sum_{k=n}^{m}\binom{m}{k}\right)^n$$

- With constant $a > 2$ one finds

$$m = an \quad\Longrightarrow\quad P_s \overset{n\to\infty}{\longrightarrow} 1$$

- Hence, the length of the linear chain required to generate an $n \times n$ cluster with $P_s$ scales as $O(n^2)$ (optimal scaling).

## References

The presented tools may even be used to asses techniques for building *redundancy encoding* resource states [10] or to prepare states rendering linear optical schemes fault tolerant [11].

[1] K. Kieling, D. Gross, J. Eisert, *JOSAB* **24** 184 (2007).
[2] D. Gross, K. Kieling, J. Eisert, *PRA* **74** 042343 (2006).
[3] K. Kieling, D. Gross, J. Eisert, quant-ph/0703045.
[4] J. Calsamiglia, N. Lütkenhaus, *APB* **72** 67 (2001).
[5] J. Eisert, *PRL* **95** 040502 (2005).
[6] R. Raussendorf, H.J. Briegel, *PRL* **86** 5188 (2001).
[7] T.B. Pittman, B.C. Jacobs, J.D. Franson, *PRA* **64** 062311 (2001).
[8] D.E. Browne, T. Rudolph, *PRL* **95** 010501 (2005).
[9] N.J.A. Sloane, The online encyclopedia of integer sequences, http://www.research.att.com/projects/OEIS?Anum=A000070.
[10] T.C. Ralph, A.J.F. Hayes, A. Gilchrist, *PRL* **95** 100501 (2005).
[11] M. Varnava, D.E. Browne, T. Rudolph, *PRL* **97** 120501 (2006).

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